International Journal of Theoretical Physics, Vol. 34, No. 2, 1995

Witten's Identity for Chern–Simons Theory

Bernd Brügmann[1]

Received June 28, 1994

We present a simple heuristic calculational scheme to relate the expectation value of Wilson loops in Chern–Simons theory to the Jones polynomial. We consider the exponential of the generator of homotopy transformations which produces the finite loop deformations that define the crossing change formulas of knot polynomials. Applying this operator to the expectation value of Wilson loops for an unspecified measure, we find a set of conditions on the measure and the regularization such that the Jones polynomial is obtained.

1. INTRODUCTION

Knot theory has recently played an important role in physics. In knot theory we study the equivalence classes of loops under diffeomorphisms connected to the identity; hence knot theory can be expected to appear naturally in the context of a diffeomorphism-invariant theory that deals with loops. Loops are of interest in any gauge theory because the trace of the parallel transport operator around a loop is a gauge invariant variable, the so-called Wilson loop. Prime examples for a diffeomorphism- and gauge-invariant theory are topological field theory (Witten, 1989) in three dimensions and canonical quantum gravity (Rovelli and Smolin, 1988) in four dimensions, and in both cases knot theory figures prominently.

The arguably most intriguing connection between knot theory and field theory is encapsulated in Witten's identity (Witten, 1989), which expresses the Jones polynomial $J_\gamma(q)$ as the vacuum expectation value of a Wilson loop $W_\gamma[A]$ in a Chern–Simons theory,

$$\langle W_\gamma[A]\rangle_{k,N} = \alpha^{w(\gamma)} J_\gamma(q) \tag{1}$$

Let us define both sides of this equation.

[1] Max-Planck-Institute of Physics, 80805 Munich, Germany. bruegman@mppmu.mpg.de.

0020-7748/95/0200-0145$07.50/0

We consider Chern–Simons theory for an $SU(N)$ connection $A_a(x) = A_a^i(x)T^i$ on a three-manifold Σ, for which we choose $\Sigma = S^3$. Here the T^i are the generators of the fundamental representation of $SU(N)$ normalized according to tr $T^iT^j = \frac{1}{2}\delta^{ij}$, and

$$\langle W_\gamma[A]\rangle_{k,N} = \int DA \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right) W_\gamma[A] \tag{2}$$

$$S_{CS}[A] = \int_\Sigma d^3x\, \epsilon^{abc} \operatorname{tr}\left(A_a\partial_b A_c + \frac{2}{3} A_a A_b A_c\right) \tag{3}$$

where $k \in \mathbb{Z}$ is the coupling constant. The Chern–Simons action has the characteristic property that

$$\frac{\delta}{\delta A_a^i(x)} S_{CS}[A] = \frac{1}{2} \epsilon^{abc} F_{bc}^i(x) \tag{4}$$

where F_{bc}^i is the curvature of A_a^i.

We define a path to be a continuous, piecewise smooth map γ: $[s, t] \to \Sigma$ with nonvanishing tangent $\dot\gamma^a(s)$, and a loop is defined as a closed path, γ: $[0, 1] \to \Sigma$ with $\gamma(0) = \gamma(1)$. Loops are allowed to have intersections (here we consider only the case of double points). A Wilson loop is defined by

$$W_\gamma[A] = \operatorname{tr} U_\gamma(0, 1) \equiv \operatorname{tr} U_\gamma \tag{5}$$

$$U_\gamma(s, t) = \mathrm{P} \exp \int_s^t du\, \dot\gamma^a(u) A_a(\gamma(u)) \tag{6}$$

where U_γ is the parallel transport operator along the loop γ, and P denotes path ordering. The Wilson loops satisfy the following differential equations:

$$\frac{\delta}{\delta A_a^i(x)} \operatorname{tr} U_\gamma = \int ds\, \dot\gamma^b(s) \delta^3(x, \gamma(s)) \operatorname{tr} U_\gamma(s) F_{ab}(\gamma(s)) \tag{7}$$

$$\frac{\delta}{\delta \gamma^a(s)} \operatorname{tr} U_\gamma = \dot\gamma^b(s) \operatorname{tr} U_\gamma(s) F_{ab}(\gamma(s)) \tag{8}$$

where $U_\gamma(s)$ is the parallel transport once around the loop from s to s.

On the right-hand side of Witten's identity (1) we have two constants that are determined by the parameters of the Chern–Simons theory,

$$q = \exp \frac{i\pi}{N + k}, \qquad \alpha = q^{N-1/N} \tag{9}$$

The whole expression is directly related to the Kauffman bracket (Kauffman, 1991) and it is *not* a diffeomorphism-invariant functional of loops. The

Kauffman bracket is defined not for knots in three dimensions, but only for knot diagrams. One usually defines invariants of knot diagrams which arise from links, i.e., from multiloops $\gamma: S^1 \times \cdots \times S^1 \to \Sigma$.

A knot diagram is a regular projection of a loop into a plane together with labels that distinguish intersections that arise from overcrossings (+) or undercrossings (−) (Fig. 1). We also introduce labels for the genuine intersection of two lines (×) and for two nonintersecting lines (≍). The unknot is denoted by (○). When discussing framing we denote a positive twist by (∝+), a negative twist by (∝−), the intermediate intersection by (∝), and the untwisted line by (ι).

The winding number $w(\gamma)$ is equal to the sum over all crossings in a knot diagram counting +1 for overcrossings and −1 for undercrossings, and it is only a regular isotopy invariant. While $w(\gamma)$ does not change under small deformations, it does depend on the projection, which may introduce an arbitrary number of crossings into the knot diagram. This type of projection dependence is also called framing dependence.

The knot polynomial $J_\gamma(q)$ is an ambient isotopy invariant and hence does not depend on the projection. It is a Laurent polynomial in one complex variable q defined by

$$J_\gamma = J_{\gamma'} \quad \text{if} \quad \gamma \sim \gamma' \quad \text{in three-space} \tag{10}$$

$$J_\circ = \frac{q^N - q^{-N}}{q - q^{-1}} \tag{11}$$

$$q^N J_+ - q^{-N} J_- = (q - q^{-1}) J_{\asymp} \tag{12}$$

The original Jones polynomial (Jones, 1985) is obtained for $N = 2$. The crossing change formula (or skein relation) (12) allows us to reduce recursively any knot diagram to the polynomial of the unknot, which is defined by (11). As usual, the crossing change formula relates polynomials for pro-

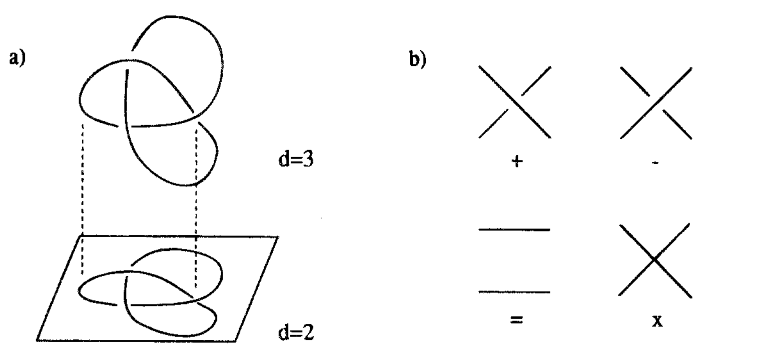

Fig. 1. (a) Projecting a knot onto a knot diagram; (b) the four crossings in a knot diagram.

jected links that differ only in one four-valent diagram, where the knot assumes one of the elementary crossings (+), (−), or (≍).

The right-hand side of Witten's identity,

$$K_\gamma(q) = \alpha^{w(\gamma)} J_\gamma(q) \tag{13}$$

is therefore the regular isotopy invariant defined by

$$K_\gamma = K_{\gamma'} \quad \text{if} \quad \gamma \sim \gamma' \quad \text{under regular isotopy} \tag{14}$$

$$K_\circ = \frac{q^N - q^{-N}}{q - q^{-1}} \tag{15}$$

$$q^{1/N} K_+ - q^{-1/N} K_- = (q - q^{-1}) K_\asymp \tag{16}$$

$$K_{\propto\pm} = \alpha^{\pm 1} K_\iota \tag{17}$$

where $w_+ - 1 = w_\asymp = w_- + 1$ and $J_{\propto\pm} = J_\iota$.

We have defined all parts of Witten's identity except for how the path integral is to be computed. In particular, we did not specify the measure DA. This is, of course, the crucial input that determines the outcome of the calculation. Let us mention four approaches toward defining the path integral.

1. First of all, there is Witten's original paper (Witten, 1989), in which he proves (1) by arguments from conformal field theory for the 2+1 decomposition of Chern–Simons theory without explicitly defining the measure DA. Witten's beautiful argument is convincing, although some of the mathematical details still have to be worked out (e.g., Schottenloher, 1993) (which applies equally well or even more so to approaches 2–4 below).

2. A quite unrelated approach is that of standard perturbation theory, in which a measure and a gauge fixing are defined explicitly (e.g., Alvarez Gaumé *et al.*, 1990; Bar-Natan, 1990; Guadagnini *et al.*, 1990; Axelrod and Singer, 1992). A proof of Witten's identity to all orders has been presented in Axelrod and Singer (1993). A recurrent problem in the perturbation theory of non-Abelian Chern–Simons theory is to reproduce the shift of k to $k + N$ in (9) (Alvarez Gaumé and Labastida, 1990; Guadagnini, 1993).

3. At the same time it was realized that geometric deformations of the loop inside the path integral are directly related to perturbative results (Smolin, 1989; Awada, 1990; Cotta-Ramusino *et al.*, 1990; Kauffman, n.d.). At least to first order in $1/k$, deformations of the loop lead to the same results as perturbation theory, and Witten's identity is satisfied to this order. Into this category falls the attempt to solve the analog of the Makeenko–Migdal loop equation known from Yang–Mills theory and thereby to obtain Witten's identity to all orders in $1/k$ (Awada, 1990).

4. An order-by-order analysis in perturbation theory shows that the coefficients in a $1/k$ expansion define particular knot invariants (Guadagnini *et*

al., 1990; Bar-Natan, 1990). These knot invariants are closely related to so-called Vassiliev invariants (Vassiliev, 1990; Baez, 1992; Bar-Natan, 1992; Kauffman, n.d.). Recently, it has been shown (Birman, 1993) that knot polynomials like the Jones polynomial arise as power series with Vassiliev invariants as coefficients. Implicit in this construction seems to be a proof that the perturbation series sums up to the Kauffman bracket.

Each of these approaches has its own merit, since the different techniques have led to different, interesting insights into the relation between Chern–Simons theory and knot theory.

The purpose of this paper is to present a formal calculational scheme based on loop deformations that allows one to derive Witten's identity to all orders in $1/k$. As such it can be viewed as an extension of the third, the geometric approach. The hope is that since $K_\gamma(q)$ is a rather simple function of loops, a simple, intuitive argument may arise by focusing on the loop dependence of $\langle W_\gamma[A]\rangle$. This is indeed the case.

Let us give a brief outline of our method. The starting point for our construction is the observation that the generator of homotopy transformations, $\tilde{D}$, applied to

$$\psi_\gamma = \langle W_\gamma[A]\rangle \tag{18}$$

reproduces the perturbation expansion to linear order in $1/k$. We define the exponential of $\tilde{D}$, which generates finite deformations of parts of a loop along a vector v. Such an operator can, for example, lift one line of a true intersection, thereby transforming $(\times)$ to $(+)$, i.e., for a suitable choice of v we have that

$$e^{\tilde{D}(v)}\psi_\times = \psi_+ \tag{19}$$

Under natural assumptions about the path integral and the regularization procedure, we derive from the calculation of $\langle \tilde{D}(v) W_\gamma[A]\rangle$ that

$$e^{\tilde{D}(v)}\psi_\times = a\psi_\times + b\psi_\asymp \tag{20}$$

for some coefficients a and b. Finally, we show that there exists an essentially unique regularization such that from (19) and (20) follow precisely the skein relations that define $K_\gamma(q)$ for $q = \exp(i\pi/k)$.

Our method must definitely be called formal, since we neither give an indirect definition of the path integral of Chern–Simons theory as in approach 1, nor an explicit definition as in approach 2. We assume that there exists an otherwise unspecified measure DA such that $\langle W_\gamma[A]\rangle$ exists. The goal is *to find a minimal set of assumptions* about the measure and regularization such that Witten's identity is reproduced.

This approach also differs from approach 4 in that we do not analyze knot invariants at each order, but only the summed series. A common feature is, however, that *loops with intersections* play a natural role. Naively, this should be expected simply because the classes of knots with intersections 'separate' the classes of nonintersecting knots. However, the great power of such a point of view in knot theory was only realized very recently (considering the long history of knot theory) in Vassiliev's work (Vassiliev, 1990; Birman, 1993). The definition of the Vassiliev invariants makes crucial use of intersecting knots, and there is a conjecture that for the first time a complete set of knot invariants may be obtained from such invariants (Birman, 1993).

In physics, a motivation to study knot invariants for loops with intersections arises in the study of the loop representation of canonical quantum gravity in 3+1 dimensions (Rovelli and Smolin, 1988, 1990). It is interesting to note that (to the knowledge of the author) an extension of the braid group to intersections appeared for the first time in Smolin (1988) and a definition of the Jones polynomial based on intersecting braids in Gambini (1992). The states in the loop representation are functionals of loops, $\psi[\eta]$, and the space of solutions to the spatial diffeomorphism constraint is the space of knot invariants. Loop functionals of noninteresting loops, however, are annihilated by the determinant of the metric [which in particular implies that they are solutions for arbitrary cosmological constant (Brügmann and Pullin, 1991)]. Nondegenerate solutions to both the diffeomorphism constraint and the Wheeler–DeWitt equation of canonical quantum gravity arise for loops with a generic triple self-intersection (Brügmann *et al.*, 1992*a*). Furthermore, these solutions are formally related to the Chern–Simons action (Brügmann *et al.*, 1992*b*), which gives rise to a solution to the constraints (Kodama, 1990) in what is known as the connection representation of quantum gravity (Ashtekar, 1991). In this context an extension of Witten's identity to intersecting loops becomes necessary, and has been given up to order $1/k$ in (Brügmann *et al.*, 1992*b*). As a by-product of our method we find that Witten's identity holds (to all orders) for the Jones polynomial of intersecting loops defined in Gambini (1992).

We will proceed as follows. In Section 2, we recall how first-order loop deformations of $\langle W_\gamma[A]\rangle$ can be computed and interpreted. In Section 3, we introduce the generator of homotopy transformations. In Section 4, we present the formal calculation that leads to Witten's identity. We conclude with a discussion in Section 5.

2. FORMAL FIRST-ORDER CALCULATION

Let us reproduce the type of calculation that shows how loop deformations in the Chern–Simons expectation value of Wilson loops lead to a first-order result that is consistent with the skein relations that define the Jones polynomial. The approach taken here is most closely related to Cotta-Ramusino *et al.* (1990) and Brügmann *et al.* (1992*b*). First we compute the functional derivative with respect to the loop of the expectation value of a single Wilson loop W_η:

$$\frac{\delta}{\delta\eta^a(s)}\langle W_\eta[A]\rangle$$

$$= \int DA \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right) \frac{\delta}{\delta\eta^a(s)} \operatorname{tr} U_\eta \tag{21}$$

$$= \dot\eta^b(s) \int DA \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right) F^j_{ab}(\eta(s)) \operatorname{tr} U_\eta(s)T^j \tag{22}$$

$$= -\frac{4\pi i}{k} \epsilon_{abc}\dot\eta^b(s) \int DA \left\{\frac{\delta}{\delta A^j_c} \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right)\right\} \operatorname{tr} U_\eta(s)T^j \tag{23}$$

$$= \frac{4\pi i}{k} \epsilon_{abc}\dot\eta^b(s) \int DA \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right) \frac{\delta}{\delta A^j_c} \operatorname{tr} U_\eta(s)T^j \tag{24}$$

$$= \frac{4\pi i}{k} \epsilon_{abc}\dot\eta^b(s) \int dt\, \dot\eta^c(t)\delta^3(\eta(s), \eta(t))$$

$$\times \int DA \exp\left(\frac{ik}{4\pi} S_{CS}[A]\right) \operatorname{tr} U_\eta(s, t)T^j U_\eta(t, s)T^j \tag{25}$$

$$= \frac{2\pi i}{k} \epsilon_{abc}\dot\eta^b(s) \int dt\, \dot\eta^c(t)\delta^3(\eta(s), \eta(t))\left(\langle \operatorname{tr} U_\eta(s, t) \operatorname{tr} U_\eta(t, s)\rangle - \frac{1}{N}\langle \operatorname{tr} U_\eta\rangle\right) \tag{26}$$

Here we have used (8), (4), and (7) in (22), (23), and (25), respectively. The last step is specific to $SU(N)$, for which

$$T^j_{ab} T^j_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right) \tag{27}$$

The pretty result is that *the loop variation of* $\langle W_\eta\rangle$ *can be again expressed in terms of expectation values of Wilson loops*. This happens precisely because the Chern–Simons action has the property (4), which is not true for a generic weight in the path integral.

Since we want to keep track of what is rigorous and what is formal, let us emphasize that two crucial assumptions have been made in (21) and (24):

(A1) The limits of differentiation and integration commute,

$$\frac{\delta}{\delta\eta^a(s)} \langle W_\eta[A] \rangle = \left\langle \frac{\delta}{\delta\eta^a(s)} W_\eta[A] \right\rangle \tag{28}$$

(A2) There are no contributions from boundary terms or the measure in partial integrations of $\delta/\delta A^j_c$,

$$\int DA \frac{\delta}{\delta A^j_c} \left\{ \exp\left(\frac{ik}{4\pi} S_{CS}[A] \right) \operatorname{tr} U_\eta(s) T^j \right\} = 0 \tag{29}$$

As we will explain below, both assumptions are wrong in general.

The result of the loop deformation can be given a rough interpretation as follows, which we will make more precise in the next section. Consider a loop η with a transverse intersection where two (but not more) lines meet, i.e., for some s and t, $s \neq t$, we have $\eta(s) = \eta(t)$ and $\dot\eta(s) \neq \dot\eta(t)$. Let us assume that the effect of the operator $v^a\delta/\delta\eta^a(s)$ is to lift one of the lines that run through the intersection along the direction v^a such that, by definition,

$$v^a \frac{\delta}{\delta\eta^a(s)} \psi_\times = \psi_+ - \psi_\times \tag{30}$$

Focusing on a single intersection, and assuming that at the point where the two loops in $\psi_\vee \equiv \langle \operatorname{tr} U_\eta(s, t) \operatorname{tr} U_\eta(t, s) \rangle$ touch we can smooth out the corners such that $\psi_\asymp \triangleq \psi_{\underset{\wedge}{\vee}}$, we can write (26) as

$$v^a \frac{\delta}{\delta\eta^a(s)} \psi_\times = \frac{2\pi i V}{k} \left(\psi_\asymp - \frac{1}{N} \psi_\times \right) \tag{31}$$

One now argues that it is a matter of proper regularization to assign a finite value to the 'volume element'

$$V = \int dt\, \epsilon_{abc} v^a \dot\eta^b(s) \dot\eta^c(t) \delta^3(\eta(s), \eta(t))$$

say $V = 1/2$.

Hence loop deformations at an intersection in the direction v^a and $-v^a$ lead to

$$\psi_+ = \left(1 - \frac{\pi i}{kN} \right) \psi_\times + \frac{\pi i}{k} \psi_\asymp \tag{32}$$

$$\psi_- = \left(1 + \frac{\pi i}{kN} \right) \psi_\times - \frac{\pi i}{k} \psi_\asymp \tag{33}$$

In order to obtain a crossing change formula for knots without intersections, we eliminate $\psi_\times$ and obtain

$$\left(1 + \frac{\pi i}{kN}\right)\psi_+ - \left(1 - \frac{\pi i}{kN}\right)\psi_- = \frac{2\pi i}{k}\,\psi_{\asymp} \tag{34}$$

In fact, this crossing change formula is to linear order in $1/k$ exactly the crossing change formula for $\psi_\gamma = K_\gamma$, (16),

$$q^{1/N}\psi_+ - q^{-1/N}\psi_- = (q - q^{-1})\psi_{\asymp} \tag{35}$$

The framing dependence of ψ_γ can be derived by a similar argument (see Section 3), and is found to agree to linear order with that of K_γ, (17).

The conclusion is that, given certain assumptions, (i) we can compute the effect of loop deformations on $\langle W_\gamma\rangle$ in 'closed' form, (ii) we can interpret the result as a skein relation for $\psi_\gamma = \langle W_\gamma\rangle$, and (iii) the coefficients in the skein relations are to linear order in $1/k$ the ones that appear in Witten's identity.

As already explained in the introduction, our goal is to explore under what conditions higher-order loop deformations lead to higher-order corrections in $1/k$ that sum to the correct coefficients. To this end, let us now give a more precise definition of the loop deformations that take intersections apart.

3. GENERATING HOMOTOPY TRANSFORMATIONS

Consider the following two generators of infinitesimal transformations of loops:

$$D(w)\psi[\eta] = \int ds\; w^a(\eta(s))\,\frac{\delta}{\delta\eta^a(s)}\,\psi[\eta] \tag{36}$$

$$\tilde{D}(v)\psi[\eta] = \int ds\; v^a(s)\,\frac{\delta}{\delta\eta^a(s)}\,\psi[\eta] \tag{37}$$

The operator $D(w)$ is the natural generator of diffeomorphisms on the space of loop functionals. Each point $\eta(s)$ of the loop is displaced an infinitesimal amount along a vector field $w^a \in T\Sigma$, and $D(w)$ satisfies the algebra of diffeomorphisms, $[D(w), D(w')] = D(\mathcal{L}_w(w'))$. Obviously, intersections of loops are invariant under the action of $D(w)$, since, independently of the parameters s and t, $\eta(s) = \eta(t)$ is moved as one single point.

On the other hand, $\tilde{D}(v)$ generates a more general type of transformation, since now v^a assigns a vector to each parameter value of the loop (as opposed to a vector to each point in the manifold). This is exactly what we need to deform loops from intersecting to nonintersecting. For example, we can choose v^a to vanish everywhere along the loop except in the neighborhood

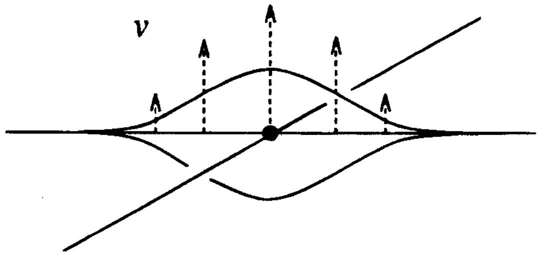

Fig. 2. A family of loops that defines a crossing change.

of *one* of the legs of an intersection such that $\eta(s)$ is moved but $\eta(t)$ remains in place.

$\tilde{D}(v)$ is the generator of homotopy transformation in loop space L. To be more precise, two loops are homotopic if they can be continuously deformed into each other. This will be the case for continuous v^a, which also can remove corners in loops. Smooth v^a lead to smooth deformations. While the orbits of $D(w)$ are curves on the three-manifold, the orbits of $\tilde{D}(v)$ are curves in L. A curve in L is defined by a one-parameter family of loops η_u, and

$$\frac{d}{du}\,\psi[\eta_u] = \int ds\, \frac{\partial \eta_u^a(s)}{\partial u}\, \frac{\delta}{\delta \eta_u^a(s)}\, \psi[\eta_u] = \tilde{D}(v)[\eta_u] \tag{38}$$

for $v^a(s) = \partial\eta_u^a(s)/\partial u$.

Our plan is to use $\tilde{D}(v)$ as a concretization of the loop deformations of the previous section. For example, we can define a family of curves that interpolates between an overcrossing and an undercrossing (with respect to some preferred direction; Fig. 2). The corresponding generator we denote by just $\tilde{D}$, and from (31),

$$\tilde{D}\psi_{\times} = \frac{\pi i}{k}\left(\psi_{\asymp} - \frac{1}{N}\,\psi_{\times}\right) \tag{39}$$

With a small modification we can also define a family of loops that stands for introducing a twist ($\propto \pm$) into a line (ι) (Fig. 3). We obtain the

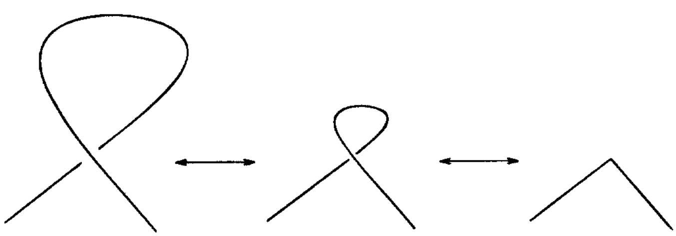

Fig. 3. A family of loops that defines the removal of a twist by changing a crossing to an intersection and simultaneously shrinking the twisted part of the loop to a point.

correct framing factor if (in inverted order) we first make the transition from crossing ($\propto$ +) to intersection ($\propto$), then shrink the loop to a corner ($<$), evaluate the path integral, and smooth the corner to (⁊). Notice that this is a different procedure than the one that defines the transition from $\psi_\propto$ to $\psi_<$ [for which the correct result is (94)]. From (26) we derive for the corresponding generator $\tilde{D}'$ that

$$\tilde{D}'\psi_< = \frac{\pi i}{k}\left(N - \frac{1}{N}\right)\psi_< \tag{40}$$

It is worth emphasizing that both generators $\tilde{D}$ and $\tilde{D}'$ refer to an essentially planar representation of a localized crossing. The family of loops for $\tilde{D}$ may only introduce a single crossing, and in this case the crossing can be made arbitrarily flat in any coordinate system. For the family of loops defining $\tilde{D}'$ we have in addition to require that the separation at the crossing and the spread in $v^a(s)$ are much smaller than the characteristic radius of the twist.

In the remainder of this section we comment on the regularization of the volume element V and on the exponentiation of $\tilde{D}$.

3.1. Regularization of the Volume Element

The volume element for $\tilde{D}(v)$ is

$$V = \int ds \int dt\ \epsilon_{abc} v^a(s)\dot{\eta}^b(s)\dot{\eta}^c(t)\ \delta^3(\eta(s), \eta(t)) \tag{41}$$

There is at least a $\delta(0)$ singularity, since the three-dimensional delta distribution depends only on two parameters.

Suppose we ignore the singularity for the moment. If we set $v^a(s) = v^a(\eta(s))$, then V is the volume element for a deformation generated by a diffeomorphism $D(v)$. Since in this case $v^a(s) = v^a(t)$, the antisymmetrization in the tangent vectors implies that $V = 0$ and therefore that $\langle W_\eta[A]\rangle$ is invariant under diffeomorphisms. We therefore expect that it is the regularization that introduces the well-known framing dependence of $\langle W_\eta[A]\rangle$.

There are two classes of singularities in V corresponding to the two classes of zeros of $\eta(s) - \eta(t)$ (in coordinates). For ranges of s and t without intersections, $\eta(s) - \eta(t) = 0$ implies $s = t$. A standard regularization in this case is to replace one of the two integrations along η by an integration around the framed loop $\eta^f(s)$, which is obtained from $\eta(s)$ by displacing $\eta(s)$ in some direction not parallel to the loop without introducing intersections. The different possibilities of framing are labeled by the linking number of the framed and unframed loops, which is an integer. Since by definition $\eta(s)$

and $\eta^f(t)$ do not intersect, we conclude that $V = 0$, and therefore that $\langle W_\eta[A]\rangle$ is invariant under diffeomorphisms when regularized by framing.

In the case that $\eta(s)$ has a self-intersection at $s_0 \neq t_0$, we have $V = 0$, as before, for a framing such that $\eta(s) \neq \eta^f(t)$ $\forall s, t$. However, we are interested in the situation where $\eta(s)$ is replaced by a family of curves $\eta_u(s)$ that describe a crossing change. In the presence of a framing, the family of curves $\eta_u(s)$ has to cross $\eta^f(t)$, and therefore there does occur a singularity for some u. This singularity is independent of the considerations that led to the framing, and we prefer to discuss it in the limit in which the framing has been removed. (Another convenient way to combine framing with crossing change is to specialize the framing such that near a self-intersection the framed loop has to maintain an intersection with the unframed loop, e.g., for a loop which is planar near an intersection the direction of displacement of the framing must be coplanar with the tangent vectors at the intersection.)

So let us consider the case where η_u describes a crossing change and the framing has been removed. The deformations we consider are along $v^a(s) = \partial\eta_u^a(s)/\partial u$ and are localized on one leg close to a crossing. Referring to a particular direction, let η_{-1} run below, η_1 above, η_0 through the intersection at $\eta_0(s_0) = \eta_0(t_0)$. These three cases correspond to ψ_-, ψ_+, and $\psi_\times$, respectively (compare Fig. 2).

In order to obtain a finite answer, we define $\psi[\eta]$ for the transition from $\psi_\times$ to ψ_+ by a 'smearing' over the strip defined by the family η_u,

$$\psi[\eta] = \int_0^1 du\, \psi[\eta_u] \tag{42}$$

Then

$$V = \int_0^1 du \int ds \int dt\, \epsilon_{abc} v^a(s)\dot\eta_u^b(s)\dot\eta_u^c(t)\delta^3(\eta_u(s), \eta_u(t)) \tag{43}$$

The delta function can be easily removed if $\epsilon_{abc}v^a(s)\dot\eta_u^b(s)\dot\eta_u^c(t) \neq 0$, since then

$$\delta^3(\eta_u(s), \eta_u(t)) = \frac{1}{|\epsilon_{abc}v^a(s_0)\dot\eta_u^b(s_0)\dot\eta_u^c(t_0)|}\delta(s - s_0)\delta(t - t_0)\delta(u - u_0) \tag{44}$$

where for our choice of v^a we obtain only one term representing the sum over all zeros of $\eta_u(s) - \eta_u(t)$.

By construction of the crossing change and by the assumption that the intersection is transverse, the tangent vectors and v^a are indeed not coplanar. Therefore the final result for the volume element when regulated by strips is in the case of an intersection

$$V = \frac{1}{2} \quad \text{for} \quad \psi_+, \qquad V = -\frac{1}{2} \quad \text{for} \quad \psi_- \tag{45}$$

as required for the first-order argument of the preceding section. [The factor 1/2 is due to $\int_0^1 dx\, \delta(x) = 1/2$.]

Since for the exponentiation of $\tilde{D}$ we have to compute $\tilde{D}^n$, we absorb the smearing (42) into the definition of $\tilde{D}$, i.e., each time $\tilde{D}$ acts we perform an extra integration over u.

3.2. Exponentiation

The generator of diffeomorphisms can be exponentiated to give finite diffeomorphisms connected to the identity. The analog is true for the generator of homotopy transformations. One way to see this is to notice that the space of loops inherits a natural differential structure as a space of mappings between two differentiable manifolds. The operator we would like to consider is

$$\exp(\tilde{D})\, \psi[\eta] \equiv \exp\left(\frac{d}{du}\right) \psi[\eta_u] \tag{46}$$

where on the left-hand side η stands for a family of loops which defines the displacement in $\tilde{D}$. Naturally, $\exp(\tilde{D})\, \psi[\eta]$ exists only if $\psi[\eta]$ satisfies an appropriate differentiability condition.

The problem is that $\psi[\eta] = [W_\eta[A]\rangle$ does not satisfy the necessary condition. Consider any function $f\colon R \to R$. Then

$$\exp\left(c\, \frac{d}{du}\right) f(u) := \sum_{n=0}^{\infty} \frac{c^n}{n!} \left(\frac{d}{du}\right)^n f(u) = f(u + c) \tag{47}$$

only if f is in fact analytic. However, we have just shown that for the regularization involving a framing that we want to use, the expectation value of a Wilson loop is a certain step function near an intersection,

$$f(u) = \langle W_{\eta_u} \rangle = \begin{cases} \psi_+ & \text{if} \quad u > 0 \\ \psi_\times & \text{if} \quad u = 0 \\ \psi_- & \text{if} \quad u < 0 \end{cases} \tag{48}$$

where ψ_+, $\psi_\times$, and ψ_- are constants independent of u.

In other words, what we would like to define is not an operator on L, but on L modulo diffeomorphisms. This is a nontrivial task, since the quotient space does not inherit a natural differential structure. A similar problem occurs in the construction of loop representations (Brügmann and Pullin, 1993), since there the loop states satisfy identities deriving from identities among the Wilson loops, which are reparametrization invariance and the

Mandelstam identities. In this context, a rigorous mathematical framework for differential calculus is being developed (Tavares, 1993), which involves an extension to distributional functionals of loops. It is still unclear whether it is possible to treat diffeomorphism invariance in this framework.

A natural point of view is that, given that $\langle W_{\eta_u} \rangle$ is a step function in u, we should have expected that d/du leads to a delta function in u. This is precisely what we obtained in Section 3.1, but there we regulated away the $\delta(u)$ by an integration over u, (42). This regularization is performed so that we can identify the first-order loop variations with the finite terms in the first-order expansion in $1/k$. We will return to this issue in Section 5.

These observations are, in fact, based on assumption (A1). In order to be able to perform the calculation (21)–(26) of the loop variations, we have to take the loop variations $\delta/\delta\eta^a(s)$ inside the path integral. In the process, we break diffeomorphism invariance. If we had independent knowledge of the step function behavior of $f(u) = \langle W_{\eta_u} \rangle$, then we could work completely on the level of distributional derivatives of $f(u)$.

The approach we take here is to motivate assumption (A1) by making an additional assumption about the regularization of $\langle W_{\eta_u} \rangle$. The problem is not to define $(d/du)\langle W_{\eta_u} \rangle$. Notice that any step function can be obtained as the limit of a sequence of analytic functions. To give a concrete example, suppose $g(u)$ is a step function that jumps from a to b at $u = 0$, and $g(0)$ is finite, e.g., $g(0) = (a + b)/2$. Then $g(u)$ is the limiting case, $\lim_{\epsilon \to 0} g_\epsilon(u) = g(u)$, of, e.g.,

$$g_\epsilon(u) = a + \frac{b - a}{\epsilon\sqrt{\pi}} \int_{-\infty}^{u} du \; e^{-x^2/\epsilon^2} \tag{49}$$

A well-defined exponential of d/du on step functions can be defined via

$$\exp\left(c \frac{d}{du}\right) g(u) := \lim_{\epsilon \to 0} \exp\left(c \frac{d}{du}\right) g_\epsilon(u) = g(u + c) \tag{50}$$

This is the definition we choose for $(d/du)\langle W_{\eta_u} \rangle$.

The assumption is that (A1) can be combined with the regularization (42) such that $(d/du)\langle W_{\eta_u} \rangle = \langle (d/du) W_{\eta_u} \rangle$. For example, for $h(u) = \exp g(u)$, which behaves under differentiation somewhat similar to $\langle (d/du) W_{\eta_u} \rangle$, this scheme is consistent.

As a consequence of the above discussion one can now show that in fact

$$\psi_{\substack{\vee \\ \wedge}} = \psi_{\asymp} \tag{51}$$

While $\tilde{D}\psi_{\substack{\vee \\ \wedge}}$ is not necessarily zero, we can define a new family of loops that describes moving the two loops in $\psi_{\substack{\vee \\ \wedge}}$ apart (rather than lifting half a leg in each of them as the original $\tilde{D}$ does). This is obvious if we choose a family of loops in the plane of the tangent vectors, since then $V = 0$. It is less obvious if we consider lifting one of the loops out of the plane of the tangent vectors, but the generator $\tilde{D}$ for this operation actually annihilates $\psi_{\substack{\vee\\ \wedge}}$, as a calculation similar to the one in Section 2 shows. Exponentiation leads to (51), if we also take into account that corners of a loop may be smoothed out, which again can be shown by an analogous argument.

4. FORMAL CALCULATION OF LOOP VARIATIONS TO ALL ORDERS AND DERIVATION OF WITTEN'S IDENTITY

In Section 3 we gave some detailed arguments that make the first-order calculation of Section 2 more rigorous; in particular, we argued that

$$\tilde{D}\psi_\times = a\psi_\times + b\psi_\asymp \tag{52}$$

for some finite coefficients a and b. Furthermore, by the construction of Section 3.2 we have that

$$e^{\tilde{D}}\psi_\times = \psi_+, \qquad e^{-\tilde{D}}\psi_\times = \psi_+ \tag{53}$$

In order to calculate the left hand side of the equations in (53), we have to know $\tilde{D}^n\psi_\times$, or $\tilde{D}\psi_\asymp$. Suppose that

$$\tilde{D}\begin{pmatrix}\psi_\times\\ \psi_\asymp\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\psi_\times\\ \psi_\asymp\end{pmatrix} \tag{54}$$

Then $(\exp \tilde{D})\psi_\times$ is known as a linear combination of $\psi_\times$ and $\psi_\asymp$, and by eliminating $\psi_\times$ in (53) we find that

$$\alpha_+\psi_+ - \alpha_-\psi_- = \alpha_\asymp\psi_\asymp \tag{55}$$

where the coefficients are functions of a, b, c, and d.

The point is that once we know that $\tilde{D}$ acts as the matrix transformation (54), the skein relation (55) follows without further assumptions about the path integral. In other words, a particular type of skein relation follows already from a generic condition on how $\tilde{D}$ acts.

The reason that we present the derivation of the skein relation in such a general manner is that it is not obvious how one should evaluate $\tilde{D}\psi_\asymp$. The main problem is that $(\exp \tilde{D})\psi_{\substack{\vee\\ \wedge}}$, if interpreted as the finite deformation of the loop as in $(\exp \tilde{D})\psi_\times$, is not the expectation value of Wilson loops. The result is $(\exp \tilde{D})\psi_{\substack{\vee\\ \wedge}} \sim \langle \mathrm{tr}\, U_\gamma\, \mathrm{tr}\, U_{\gamma'}\rangle + \ldots$, where γ and γ' are open paths. In particular, the trace of the parallel transport along an open path is not gauge invariant.

Let us therefore turn things around and ask whether there exists matrices $\tilde{D}$ at all such that we obtain Witten's result, i.e., that

$$\alpha_+ = q^{1/N}, \qquad \alpha_- = q^{-1/N}, \qquad \alpha_\asymp = q - q^{-1} \tag{56}$$

It is important to notice that if so, this is a nontrivial statement, since, as we will see below, arbitrary choices of $\tilde{D}$ do not generate all possible skein relations.

Instead of determining the matrix $\tilde{D}$ that leads to the 'correct' skein relation directly from the explicit formulas that exist for the exponential of a 2×2 matrix, which are subject to various conditions on the matrix, it is simpler and perhaps more intuitive to fix $\tilde{D}$ by an order-by-order analysis in an expansion in $1/k$ and then to check whether exponentiation gives the right result to all orders.

The outcome of such an analysis is that for

$$q = \exp \frac{\pi i}{N + k} \tag{57}$$

there does not exist a matrix $\tilde{D}$ that leads to Witten's result. (The adaptation of the coefficients fails at order $1/k^3$.) This has to be expected, since each power of $\tilde{D}$ produces an overall factor of $\pi i/k$, and it is hard to see how these coefficients can combine to $\pi i/(N + k)$, although *a priori* the possibility of a suitable 'resummation' cannot be excluded.

There does exist a matrix $\tilde{D}$ that leads to Witten's result without the shift in k, i.e., for

$$q = \exp \frac{\pi i}{k} \tag{58}$$

It is a well-known problem of certain approaches to perturbative Chern–Simons theory (Guadagnini *et al.*, 1990; Awada, 1990; Bar-Natan, 1990) that they do not reproduce the shift $k \mapsto N + k$, and our method seems to be of the same type. It is sometimes argued that the shift is due to an independent argument about the existence of the path integral—this is consistent since k appears only in one place (in the definition of q)—or that there is no need for the shift (Guadagnini, 1993).

Let us sketch the calculation. We explicitly compute the matrix $\tilde{D}^n$ to find

$$\psi_+ = \sum_{n=0}^{m} \frac{1}{n!} \tilde{D}^n \psi_\times + O\left(\frac{1}{k^{m+1}}\right) \tag{59}$$

from which we obtain the coefficients α_+, α_-, and $\alpha_\asymp$ in (55) up to order m. We try to choose a, b, c, and d such that these coefficients agree order by order in $1/k$ with (56) for

$$q^x = \sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\pi i}{k}x\right)^n + O\left(\frac{1}{k^{m+1}}\right) \tag{60}$$

for some exponent x. We find that

$$\text{first order} \Rightarrow a = -\frac{\pi i}{kN}, \quad b = \frac{\pi i}{k} \tag{61}$$

$$\text{second order} \Rightarrow c = 0 \tag{62}$$

$$\text{third order} \Rightarrow d = \frac{\pi i}{k}\left(\pm 1 - \frac{1}{N}\right) \tag{63}$$

The two choices for d turn out to be equivalent, so we pick the plus sign. This fixes the available freedom.

Since

$$\exp\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e^a & (e^a - e^b)/(a - b) \\ 0 & e^b \end{pmatrix} \tag{64}$$

for $a \neq b$ (see below), we have that

$$e^{\tilde{D}}\psi_{\times} = q^{-1/N}\psi_{\times} + q^{-1/N}(q - 1)\psi_{\asymp} \tag{65}$$

and the corresponding equation for $\exp(-\tilde{D})$ from $q \mapsto q^{-1}$. The end result is the correct skein relation, (16),

$$q^{1/N}\psi_{+} - q^{-1/N}\psi_{-} = (q - q^{-1})\psi_{\asymp} \tag{66}$$

To arrive at Witten's identity, we also have to derive the framing dependence of $\psi_\gamma = \langle W_\gamma[A]\rangle$. The appropriate generator, $\tilde{D}'$, was already defined in Section 3. By definition,

$$e^{\tilde{D}'}\psi_{<} = \psi_{\propto +} \tag{67}$$

Since $\tilde{D}'\psi_{<}$ is proportional to $\psi_{<}$, exponentiation is trivial and we obtain from (40) that

$$\psi_{\propto +} = \alpha\psi_{\iota}, \qquad \psi_{\propto -} = \alpha^{-1}\psi_{\iota} \tag{68}$$

$$\alpha = q^{N-1/N} \tag{69}$$

which is the correct framing relation, (17).

What we have to show in order to derive Witten's identity is that

$$\psi_\gamma = K_\gamma \tag{70}$$

where K_γ is defined by (14)–(17). We just established the skein relation (16) and the framing relation (17). That ψ_γ is a regular isotopy invariant, (14), follows from Section 3.1, where we argued that ψ_γ is an invariant of framed loops. The normalization of ψ_γ corresponds to a normalization of the path integral and can be freely specified as in (15).

This concludes the formal derivation of Witten's identity. Linear order considerations led us to assumptions (A1) and (A2). If we insist that the exponentiation of $\tilde{D}$ leads to Witten's identity in the particular scheme that we introduced, then we are forced at third order to make the following assumption:

(A3) We have

$$\tilde{D}\psi_{\overset{\vee}{\wedge}} = \frac{\pi i}{k}\left(1 - \frac{1}{N}\right)\psi_{\overset{\vee}{\wedge}} \tag{71}$$

which we interpret as an assumption about regularization. The simplicity of the proposal lies in the fact that (A1)–(A3) reduce the derivation of Witten's identity to essentially three functional integrations with respect to η^a and A_a^i and the exponentiation of a 2×2 matrix.

5. DISCUSSION

Let us discuss (A1)–(A3) and comment on two applications of our method.

- On (A1). In Section 3.2 we argued that to commute the limits of differentiation and integration makes sense for $(d/du)\langle W_{\eta_u}\rangle$ if we smooth out the step function in u. There are no conclusive arguments why one should regulate this way, but the construction seems natural enough.

- On (A2). For a generic definition of the measure DA and the boundary conditions, the assumption that in a partial integration there are no extra terms is certainly wrong. It would seem that this issue cannot be discussed without invoking the machinery of standard perturbation theory to explicitly define a measure and a gauge fixing. However, very recently Ashtekar and Lewandowski (n.d.) constructed explicitly a diffeomorphism- and gauge-invariant measure on $\overline{\mathcal{A}/SU(N)}$, i.e., on a suitable completion of the space of connections modulo gauge transformations involving distributions [see Baez (1993) for an extension]. This may come as a surprise and be of some importance for the path integral formulation, since, as generally believed, there do not exist such measures on $\mathcal{A}/SU(N)$ (which is neither Hausdorff nor locally compact).

Although the Ashtekar–Lewandowski measure $(dA)_{AL}$ is not the correct measure for our purposes, we consider the fact that such measures do exist as encouragement that naive manipulations of the path integral may make sense after all, in particular without gauge fixing. For example:

(i) Notice that a regulated version of

$$\hat{T}^a[\eta](s) = \operatorname{tr} U_\eta(s) \frac{\delta}{\delta A_a(\eta(s))} \tag{72}$$

is a self-adjoint operator with respect to $(dA)_{\mathrm{AL}}$ (Ashtekar and Isham, 1992; Ashtekar and Lewandowski, n.d.). $\hat{T}^a$ is the generator of diffeomorphisms (plus an irrelevant gauge transformation) in the connection representation of canonical Yang–Mills theory (Ashtekar, 1992). Let us define the Chern–Simons path integral in terms of $(dA)_{\mathrm{AL}}$ and assume that the integrand is integrable. Wilson loops are integrable, but it is not clear that $\exp[(ik/4\pi)S_{\mathrm{CS}}]$ is integrable (which may make a regularization and renormalization necessary). Then in the calculation of loop variations in Section 2 we have that

$$\int (dA)_{\mathrm{AL}} \exp\left(\frac{ik}{4\pi} S_{\mathrm{CS}}\right) F^j_{ab} \operatorname{tr} U_\eta(s) T^j$$

$$= -\frac{4\pi i}{k} \int (dA)_{\mathrm{AL}}\, \epsilon_{abc} \hat{T}^c[\eta](s) \exp\left(\frac{ik}{4\pi} S_{\mathrm{CS}}\right) \tag{73}$$

$$= 0 \tag{74}$$

since $\hat{T}^a$ is self-adjoint (and $\hat{T}^a 1 = 0$). That the result is zero is actually consistent with the definition of $(dA)_{\mathrm{AL}}$, for which the integral over a Wilson loop is zero if the loop is traversed an odd number of times, so that also the left-hand side of (73) is expected to be zero. Hence, in this scenario partial integration is well defined and trivial in the sense of (A1), although the result, while diffeomorphism invariant, is not Witten's identity.

(ii) Suppose that DA is a diffeomorphism-invariant measure such that the Chern–Simons expectation value exists. Recall the discussion of Section 3.1, where we argued for regulating the $\delta(u)$ singularity. Suppose now that we do not smear over u and decide to deal with the derivative of the step function directly. Then since

$$\psi_+ - \psi_\times = \int_0^1 du \frac{d}{du} \psi[\eta_u] \tag{75}$$

we would conclude from (39) that

$$\psi_+ - \psi_\times = \frac{\pi i}{k}\left(\psi_{\asymp} - \frac{1}{N}\psi_\times\right) \tag{76}$$

is *exact*, i.e., satisfied without approximation in $1/k$. The obvious conclusion is that the necessary corrections arise from the partial integration. The term that we have neglected is

$$-\frac{4\pi i}{k}\int_0^1 du \int ds\, \epsilon_{abc} v^a(s)\dot{\eta}_u^b(s) \int DA\, \frac{\delta}{\delta A_c^j}\left[\exp\left(\frac{ik}{4\pi} S_{\rm CS}\right) \operatorname{tr} U_{\eta_u}(s)T^j\right] \quad (77)$$

If we assume that $DA = dA\ \mu(A)$, where $\delta/\delta A_c^j$ is self-adjoint with respect to dA, then we can obtain an identity for the weight $\mu(A)$. For example, if

$$\epsilon_{abc}\frac{\delta}{\delta A_c^j}\,\mu(A) = F_{ab}^j\mu(A) \quad (78)$$

we obtain a series in $1/k$, which is the feature we are looking for, but the result is wrong for two reasons. Witten's identity cannot be obtained from $\psi_{\asymp} - (1/N)\psi_{\times}$ times an overall factor, and we need a factor $1/n!$ for an exponential. If such an iteration is successful, one should also recover the shift.

Suppose we take the point of view that there exists a diffeomorphism-invariant measure similar to $(dA)_{\rm AL}$ for which (A1) and (A2) can be made rigorous by certain corrections. Then the curious fact remains that the 'mistakes' we make by assuming (A1) and (A2) can be compensated by (A3).

• On (A3). The result that there is a matrix $\tilde{D}$ as in (54) that leads to Witten's identity is a nontrivial feature of our method since (54) cannot accommodate arbitrary skein relations. Let us look at a representative example for generic $\tilde{D}$. For 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it is simple to compute e^A, since

$$A = C\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}C^{-1} \Rightarrow e^A = C\begin{pmatrix} e^{\lambda}_1 & 0 \\ 0 & e^{\lambda}_2 \end{pmatrix}C^{-1} \quad (79)$$

if C exists. If $ad - bc \neq 0$ and $(a-d)^2 + 4bc > 0$, then a possible choice is

$$C = \begin{pmatrix} \lambda_1 - d & b \\ c & \lambda_2 - a \end{pmatrix}, \qquad C^{-1} = \frac{1}{\det C}\begin{pmatrix} \lambda_2 - a & -b \\ -c & \lambda_1 - d \end{pmatrix} \quad (80)$$

$$\lambda_{1,2} = \frac{1}{2}\{a + d \pm [(a-d)^2 + 4bc]^{1/2}\} \quad (81)$$

Hence,

$$\alpha_+ = (a_1 e^{-\lambda_1} - a_2 e^{-\lambda_2})/\det C \quad (82)$$

$$\alpha_- = (a_1 e^{\lambda_1} - a_2 e^{\lambda_2})/\det C \quad (83)$$

$$\alpha_{\asymp} = b_1(a_1 - a_2)(e^{\lambda_2 - \lambda_1} - e^{\lambda_1 - \lambda_2})/(\det C)^2 \quad (84)$$

where $a_1 = (\lambda_1 - d)(\lambda_2 - a)$, $a_2 = bc$, and $b_1 = b(\lambda_1 - d)$.

Notice that for such A, one does not in general obtain a skein relation of the general type (Cotta-Ramusino *et al.*, 1990)

$$\beta\psi_+ - \beta^{-1}\psi_- = z\psi_{\asymp} \tag{85}$$

for some complex coefficients β and z, since in general $\alpha_+ \neq \alpha_-^{-1}$. At the same time, the coefficients cannot be made arbitrary (even if A is more special).

Let us address the question of whether there is a natural interpretation for the particular choice of $\tilde{D}\psi_\times$ made in (A3). There are, in fact, two natural suggestions for $\tilde{D}\psi_\times$, which, however, lead to the wrong result. One proposal that unfortunately fails is to just repeat the calculation of $\tilde{D}\psi_\times$ of Section 2 for $\tilde{D}(\tilde{D}\psi_\times)$. The result is

$$(\tilde{D}_1)^2\psi_\times := \tilde{D}(\tilde{D}\psi_\times) \sim \frac{1}{2}\langle \mathrm{tr}\ U_\gamma T^i T^j U_{\gamma'} T^i T^j + \mathrm{tr}\ U_\gamma T^i T^j U_{\gamma'} T^j T^i \rangle \tag{86}$$

From this it follows (or directly from $\tilde{D}\psi_{\substack{\vee\\ \wedge}}$ with due care about the corners) that

$$\tilde{D}_1 \begin{pmatrix} \psi_\times \\ \psi_{\asymp} \end{pmatrix} = \frac{\pi i}{k} \begin{pmatrix} -1/N & 1 \\ 1/2 & N/2 - 1/N \end{pmatrix} \begin{pmatrix} \psi_\times \\ \psi_{\asymp} \end{pmatrix} \tag{87}$$

This matrix is of the general type that we just analyzed, but the resulting skein relation is not even that of a knot polynomial of the type (85). There is no obvious reason why one should subtract out the terms leading to (A3).

Another idea could be to postulate that

$$\tilde{D}_2\psi_{\substack{\vee\\ \wedge}} = 0 \tag{88}$$

This amounts to mixing the limit in which $\psi_{\substack{\vee\\ \wedge}} = \psi_{\asymp}$ with the limit of differentiation, the idea being that $\tilde{D}_2\psi_{\asymp}$ is zero because the loops are separated. Previously we smoothed out the corners always after all derivatives are taken. The resulting skein relation is

$$q^{1/N}\psi_+ - q^{-1/N}\psi_- = N(q^{1/N} - q^{-1/N})\psi_{\asymp} \tag{89}$$

which does fit the definition of a knot polynomial, although not Witten's identity. The resulting framing relations are not of the type $\psi_{\alpha\pm} = \alpha^{\pm 1}\psi$. Notice that $c = 0$ implies skein relations of the type (85), while the choice for d given by (A3) is distinguished, since then $\alpha_{\asymp}$ is independent of N.

If one decides to introduce a relative factor unequal to one in $\psi_{\substack{\vee\\ \wedge}} \sim \psi_{\asymp}$ that adjusts (89) to the correct result (16), then one runs into inconsistent framing relations. A simple check of the self-consistency of the skein and framing relations are given by the Mandelstam identities. For example, for $SU(2)$ matrices A and B, $\mathrm{tr}\ A\ \mathrm{tr}\ B = \mathrm{tr}\ AB + \mathrm{tr}\ AB^{-1}$, which leads to some simple consequences that are correct for $\tilde{D}$ but incorrect for $\tilde{D}_2$ with the rescaling.

• Abelian Chern–Simons theory. Assumptions (A1) and (A2) and the regularization may be explicitly testable in Abelian Chern–Simons theory. The relevant equations in this case are

$$\psi_{\times} = \psi_{\substack{\vee \\ \wedge}} \tag{90}$$

$$\tilde{D}\psi_{\times} = \frac{2\pi i}{k}\,\psi_{\times} \tag{91}$$

$$\psi_{\pm} = q^{\pm 2}\psi_{\asymp} \tag{92}$$

Notice that the last equation implies the well-known result that all the information of ψ_γ is in $w(\gamma)$. In particular, it would be interesting to find out whether there exists a direct relation to perturbation theory (e.g., Coste and Makowka, 1990).

• Skein relations for intersections. As mentioned in the introduction, the extension of knot invariants to intersecting loops is of interest in quantum gravity. From (65) we obtain immediately the skein and 'framing' relations for intersections,

$$q^{1/N}(1 - q^{-1})\psi_{+} - q^{-1/N}(1 - q)\psi_{-} = (q - q^{-1})\psi_{\times} \tag{93}$$

$$\psi_{\propto} = \frac{q^{N}(1 - q^{-1}) - q^{-N}(1 - q)}{q - q^{-1}}\,\psi_{\iota} \tag{94}$$

Independent of Witten's identity, a two-variable version of the Jones polynomial for intersecting loops, $F_\gamma(q, a)$, can be constructed from the braid group with intersections (Gambini, 1992; Brügmann *et al.*, 1992*b*). Skein relation (93) corresponds for $N = 2$ precisely to the choice $a = 1 - q^{-1}$ in $F_\gamma(q, a)$. The functional $\psi_\times$ is therefore well defined in its own right, and it is not just a meaningless variable that is introduced for purely technical reasons and that has to be eliminated from the relation between ψ_+, ψ_-, $\psi_{\asymp}$, and $\psi_\times$.

What we also would like to point out is that these relations are nontrivial extensions of the linear order results, and have to be contrasted with the defining relations of the Vassiliev invariants. [Their relationship is discussed in Baez (1992) and Kauffman (n.d.).] To linear order we have $\psi_\times = \frac{1}{2}(\psi_+ + \psi_-) + O(1/k^2)$ and $\psi_{\propto} = \psi_{\iota} + O(1/k^2)$. On the other hand, the characteristic relation for intersections of the Vassiliev invariants is $V_\times = V_+ - V_-$ (no k dependence).

In conclusion, loop deformations in $\langle W_\gamma[A]\rangle$ allow us to derive skein relations for $\langle W_\gamma[A]\rangle$ that are to linear order in $1/k$ that of K_γ. The necessary assumptions (A1) and (A2) are supported by results in perturbation theory, and by the fact that we obtain the correct answer. A novel aspect of the construction is that we define a generator of loop deformations that can be exponentiated. The final result depends on assumption (A3) for $\tilde{D}\psi_{\vee}$, which is perhaps less well founded than (A1) and (A2), but gives the correct answer to all orders. What we would like to find is some further evidence for (A3), say from an explicit regularization.

The heuristic level of our discussion does not allow us to decide whether a rigorous proof of Witten's identity can be constructed from (A1)–(A3). On the other hand, we find it remarkable that all the uncertainty about the measure *DA* can be condensed into such a simple set of assumptions.

ACKNOWLEDGMENTS

It is a pleasure to thank P. Hájiček, A. Higuchi, M. Niedermaier, and J. Pullin for interesting discussions.

REFERENCES

Alvarez Gaumé, L., and Labastida, J. (1990). *Nuclear Physics B (Proceedings Supplement)* **18B**, 1–9.

Alvarez Gaumé, L., Labastida, J., and Ramallo, A. (1990). *Nuclear Physics B*, **334**, 103–24.

Ashtekar, A. (1991). *Lectures on Non-Perturbative Canonical Gravity*, World Scientific, Singapore.

Ashtekar, A., and Isham, C. (1992). *Classical and Quantum Gravity*, **9**, 1433–1485.

Ashtekar, A., and Lewandowski, J. (n.d.). Representation theory of analytic holonomy C^* algebras, in *Knots and Quantum Gravity*, J. Baez, ed., Oxford University Press, Oxford, in press.

Awada, M. (1990). *Communications in Mathematical Physics*, **129**, 329–349.

Axelrod, S., and Singer, I. (1992). Chern–Simons perturbation theory, in *Proceedings of the XXth DGM Conference*, S. Catto and A. Rocha, eds., World Scientific, Singapore, pp. 3–45.

Axelrod, S., and Singer, I. (1993). Chern–Simons perturbation theory II, MIT preprint, hep-th/9304087.

Baez, J. C. (1992). *Letters on Mathematical Physics*, **26**, 43–51.

Baez, J. C. (n.d.). Diffeomorphism-invariant generalized measures on the space of connections modulo gauge transformations, in the proceedings of the Conference on Quantum Topology, Manhattan, Kansas, March 1993, to appear.

Bar-Natan, D. (1990). Perturbative Chern–Simons theory, Princeton preprint (August 1990).

Bar-Natan, D. (1992). On the Vassiliev knot invariants, Princeton preprint (August 1992).

Birman, J. (1993). *Bulletin of the American Mathematical Society*, **28**, 253–287.

Brügmann, B., and Pullin, J. (1991). *Nuclear Physics B*, **363**, 221–244.

Brügmann, B., and Pullin, J. (1993). *Nuclear Physics B*, **390**, 399–438.

Brügmann, B., Gambini, R., and Pullin, J. (1992*a*). *Physical Review Letters*, **68**, 431–434.

Brügmann, B., Gambini, R., and Pullin, J. (1992*b*). *Nuclear Physics B*, **385**, 587–603.

Coste, A., and Makowka, M. (1990). *Nuclear Physics B*, **342**, 721–736.

Cotta-Ramusino, P., Guadagnini, E., Martellini, M., and Mintchev, M. (1990). *Nuclear Physics B*, **330**, 557–574.
Gambini, R. (1992). Link invariant polynomials for intersecting loops, Montevideo preprint IFFI.
Guadagnini, E. (1993). *The Link Invariants of the Chern–Simons Field Theory*, De Gruyter.
Guadagnini, E., Martellini, M., and Mintchev, M. (1990). *Nuclear Physics B*, **330**, 575–607.
Jones, V. (1985). *Bulletin of the American Mathematical Society*, **12**, 103–110.
Kauffman, L. (1991). *Knots and Physics*, World Scientific, Singapore.
Kauffman, L. (n.d.). Vassiliev invariants and the loop states in quantum gravity, in *Knots and Quantum Gravity*, J. Baez, ed., Oxford University Press, Oxford, in press.
Kodama, H. (1990). *Physical Review D*, **42**, 2548–2565.
Rovelli, C., and Smolin, L. (1988). *Physical Review Letters*, **61**, 1155–1158.
Rovelli, C., and Smolin, L. (1990). *Nuclear Physics B*, **331**, 80–152.
Schottenloher, M. (1993). Metaplectic quantization of the moduli spaces of flat and parabolic bundles (after P. Scheinost), LMU preprint.
Smolin, L. (1988). Knot theory, loop space and the diffeomorphism group, in *New Perspectives in Canonical Gravity*, A. Ashtekar with invited contributions, Bibliopolis, pp. 245–266.
Smolin, L. (1989). *Modern Physics Letters A*, **4**, 1091–1112.
Tavares, J. (1993). Chen integrals, generalized loops and loop calculus, Preprint, University of Porto (April 1993).
Vassiliev, V. A. (1990). Cohomology of knot spaces, in *Theory of Singularities and Its Applications*, V. Arnold, ed., *Advances in Soviet Mathematics*, Vol. 1, pp. 23–69.
Witten, E. (1989). *Communications in Mathematical Physics*, **121**, 351–359.